J. Math. Pures Appl.,
76, 1997, *p.* 125 *à* 153

DIRICHLET QUOTIENTS AND 2D PERIODIC NAVIER-STOKES EQUATIONS

By Peter CONSTANTIN, Ciprian FOIAS, Igor KUKAVICA and Andrew J. MAJDA

ABSTRACT. – We show that for the periodic 2D Navier-Stokes equations (NSE) the set of initial data for which the solution exists for all negative times and has exponential growth is rather rich. We study this set and show that it is dense in the phase space of the NSE. This yields to a positive answer to a question in [BT]. After an appropriate rescaling the large Reynolds limit dynamics on this set is Eulerian.

Key words and phrases: Navier-Stokes equations, Dirichlet quotients, Euler equation.

1. Introduction and motivation

Many partial and ordinary differential equations connected with fluid dynamics have the structure

$$\dot{u} + \nu A u + B(u,u) = f, \tag{1.1}$$

where A is a closed operator on a suitable Hilbert space H, $B(\cdot,\cdot)$ is a bilinear form, and $f \in H$ is time independent. Examples of such equations are the Navier-Stokes equations (NSE), the Kuramoto-Sivashinsky equation, and the Ginzburg-Landau equation (*see e.g.* [T2]).

Naturally, the simplest special case is when the nonlinear term B vanishes. In this case the spectral properties of A are intimately connected to the stable manifolds of (1.1). More precisely, assume for simplicity that A is a closed positive linear operator with a compact inverse, and let $0 < \lambda_1 < \lambda_2 < \cdots$ be its distinct eigenvalues. Then, for every $u_0 \in H$,

$$u(t) = S(t)u_0 = e^{-\nu t A}\left(u_0 - \frac{1}{\nu}A^{-1}f\right) + \frac{1}{\nu}A^{-1}f$$

is the solution of (1.1) with $B = 0$ such that $u(0) = S(0)u_0 = u_0$. Note that, for every $u_0 \in H$, $S(t)u_0$ converges to the global attractor $\mathcal{A} = \{A^{-1}f\}$. If $|\cdot|$ denotes the norm in the Hilbert space H, we define

$$\mathcal{M}_n = \mathcal{A}\bigcup\left\{u_0 \in H\backslash\mathcal{A} : \limsup_{t\to-\infty}\frac{|A^{1/2}S(t)u_0|^2}{|S(t)u_0|^2} \le \frac{\lambda_n + \lambda_{n+1}}{2}\right\}. \tag{1.2}$$

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. – 0021-7824/1997/02/\$ 7.00

We emphasize that $S(t)u_0$ is required to be defined for all $t < 0$. Then, for every $n \in \mathbb{N}$,

$$\mathcal{M}_n = \left\{ u_0 \in H : u_0 - \frac{1}{\nu} A^{-1} f \in P_n H \right\},$$

where P_n denotes the orthogonal projector on the spectral space of A associated with $\{\lambda_1, \ldots, \lambda_n\}$. Note that $\mathcal{M}_n$ consists precisely of those solutions $S(t)u_0$ which exist for all $t \in \mathbb{R}$ and whose norm increases slower than const $\cdot\, e^{\nu\lambda_n|t|}$ as $t \to -\infty$, *i.e.*,

$$\mathcal{M}_n = \left\{ u_0 \in H : |S(t)u_0| = \mathcal{O}(e^{\nu\lambda_n|t|}) \text{ as } t \to -\infty \right\}. \tag{1.3}$$

The definitions (1.2) or (1.3) make sense for any equation of the form (1.1) where $S(t)$ represents the solution map. However, in general, all one can expect is that $\mathcal{M}_n$ contains the global attractor. Indeed, one can prove that for the the Kuramoto-Sivashinsky equation and Ginzburg-Landau equation $\mathcal{M}_n = \mathcal{A}$, *i.e.*, if a solution grows at most exponentially as $t \to -\infty$, then it is necessarily uniformly bounded (*see e.g.* [K] and [DGHN]). It is thus quite remarkable that for the periodic 2D Navier-Stokes equations the situation is much closer to the linear case.

In the present paper we develop a theory regarding the invariant sets $\mathcal{M}_n$. In Section 3 we show that the sets $\mathcal{M}_n$ are rather rich: We prove that $P_n\mathcal{M}_n = P_nH$ for every n. Obviously, this implies that the Hausdorff (and hence also the fractal) dimension of the set $\mathcal{M}_n$ intersected with any ball in H is at least dim P_nH. We give a positive answer to the following question of Bardos and Tartar ([BT]): Is $S(t)H$ dense for a fixed $t > 0$? We prove that even more $\bigcap_{t\geq 0} S(t)H$ is dense, however in a weaker topology. In Section 4 we study the behavior of Dirichlet quotients $|A^{1/2}u(t)|/|u(t)|$ for solutions u of the NSE as $t \to -\infty$ and use them to obtain a precise rate of exponential growth of solutions as $t \to -\infty$. In Section 5 we study the dynamics on the invariant sets $\mathcal{M}_n$. We show that the normalized solutions $u(t)/|u(t)|$ lead, as $t \to -\infty$, to global solutions of the incompressible Euler equation. This enables us to introduce attractor-type sets which are invariant with respect to the Euler equation and attract the quotients $u(t)/|u(t)|$ when $t \to -\infty$ for $u(0) \in \mathcal{M}_n \backslash \mathcal{A}$. In Section 6 we discuss another density type property of the invariant sets $\mathcal{M}_n$. We conclude the paper with a list of open problems.

2. Functional form of the NSE and some known facts

We consider the Navier-Stokes equations (NSE) on $\Omega = [0, L]^2$

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f,$$

$$\nabla \cdot u = 0,$$

$$u, p \ \ \Omega\text{-periodic}, \qquad \int_\Omega u = 0,$$

where $u: \mathbb{R}^2 \to \mathbb{R}^2$, $p: \mathbb{R}^2 \to \mathbb{R}$ are unknown functions, and $\nu > 0$, $L > 0$, and $f \in L^2(\Omega)^2$ (which is Ω-periodic and $\int_\Omega f = 0$) are given. We introduce spaces H and V as the closures of

$$\left\{ v \in L^2(\Omega)^2 : v \text{ is an } \Omega\text{-periodic trigonometric polynomial}, \nabla \cdot v = 0 \text{ in } \Omega, \int_\Omega v = 0 \right\}$$

in the (real) Hilbert spaces $L^2(\Omega)^2$ and $H^1(\Omega)^2$, respectively. The sets H and V are also Hilbert spaces with respective scalar products

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j v_j \;, \qquad u, v \in H$$

and

$$((u, v)) = \sum_{j,k=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_k} \frac{\partial v_j}{\partial x_k} \,, \quad u, v \in V.$$

The corresponding norms are $|u| = (u, u)^{1/2}$, for $u \in H$, and $\|u\| = ((u, u))^{1/2}$, for $u \in V$. By the Rellich imbedding theorem, the natural inclusions $i_1: V \to H$ and $i_2: H \to V'$ are compact, V' being the dual of V.

Let $P_L: L^2(\Omega)^2 \to L^2(\Omega)^2$ be the orthogonal projection (called the Leray projector) with the range H, and let $A = -P_L \Delta$ be the Stokes operator, which is a positive operator in H with the domain of definition $D(A) = H^2(\Omega)^2 \bigcap V$. Introducing $B(u, v) = P_L((u \cdot \nabla)v)$ and $\dot{u} = \partial u / \partial t$, the NSE can be written as

$$\dot{u} + \nu A u + B(u, u) = f, \tag{2.1}$$

where we replaced $P_L f$ with f. This equation is the functional form of the NSE, and it is understood in V'.

Classical theorems imply that, for every $u_0 \in H$, (2.1) has a unique solution $u(t) = S(t)u_0$ for $t \geq 0$, which satisfies $u(0) = u_0$ and $u \in C_b([0, \infty), H) \cap C_{\text{loc}}((0, \infty), V) \cap L^2_{\text{loc}}([0, \infty), V)$. (We always assume $f \in H$.) If the solution $u(t)$ exists also for $t \in [-t_0, 0]$, where $t_0 > 0$, then it is still uniquely determined by $u_0 = u(0)$ (*see* [BT] or [CF, Theorem 12.2]); therefore, we still denote its value $u(t)$ at $t \in [-t_0, 0]$ by $S(t)u_0$. Also, for any $t_0 \geq 0$, the solution operator $S(t_0): H \to H$ is continuous; similarly, $S(t_0): V \to V$ is continuous.

Now, we recall some spectral properties of A. First, A is a positive operator with eigenvalues $(k_1^2 + k_2^2)(2\pi/L)^2$ where $k_1, k_2 \in \mathbb{N}$ and $k_1^2 + k_2^2 \neq 0$. We arrange them in increasing order

$$\left(\frac{2\pi}{L}\right)^2 = \lambda_1 < \lambda_2 < \cdots.$$

In particular, the identity $\|u\| = |A^{1/2}u|$ ($u \in V$) implies the Poincaré inequality

$$\|u\|^2 \geq \lambda_1 |u|^2 \,, \qquad u \in V. \tag{2.2}$$

An important property regarding the spectral gaps is

$$\limsup_{n\to\infty}(\lambda_{n+1}-\lambda_n)=\infty, \tag{2.3}$$

(*see* [E] or [R]). Also, note that

$$\lambda_{n+1}-\lambda_n \geq \left(\frac{2\pi}{L}\right)^2 = \lambda_1 , \qquad n\in\mathbb{N}$$

and

$$\lim_{n\to\infty}\lambda_n=\infty. \tag{2.4}$$

Regarding the bilinear form B, we will need the inequalities

$$|(B(u,v),w)| \leq C|u|^{1/2}\,\|u\|^{1/2}\,\|v\|\,|w|^{1/2}\,\|w\|^{1/2} , \qquad u,v,w\in V \tag{2.5}$$

and

$$|A^{-1/2}B(u,v)| \leq C|u|^{1/2}\,\|u\|^{1/2}\,|v|^{1/2}\,\|v\|^{1/2} , \qquad u,v\in V. \tag{2.6}$$

Also, we will use the identities

$$(B(u,u),u)=0 , \qquad u\in V \tag{2.7}$$

and

$$(B(u,u),Au)=0 , \qquad u\in D(A). \tag{2.8}$$

Both can be obtained using integration by parts (*see e.g.* [CF]). The identity (2.7) implies

$$\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2=(f,u)\leq\frac{1}{2\nu\lambda_1}|f|^2+\frac{\nu\lambda_1}{2}|u|^2 \tag{2.9}$$

which, because of (2.2), shows that

$$\frac{d}{dt}|u|^2+\nu\lambda_1|u|^2\leq\frac{1}{\nu\lambda_1}|f|^2. \tag{2.10}$$

If u is defined on some interval $[t_0,\infty)$, the Gronwall lemma gives

$$|u(t)|^2\leq|u(t_0)|^2e^{-\nu\lambda_1(t-t_0)}+\frac{|f|^2}{\nu^2\lambda_1^2}\left(1-e^{-\nu\lambda_1(t-t_0)}\right) , \qquad t\geq t_0 ; \tag{2.11}$$

also,

$$|u(t)|^2\geq|u(t_0)|^2e^{\nu\lambda_1(t_0-t)}-\frac{|f|^2}{\nu^2\lambda_1^2}\left(e^{\nu\lambda_1(t_0-t)}-1\right) , \qquad t\leq t_0, \tag{2.12}$$

provided u is defined on $[t,t_0]$. The inequality (2.11) shows that $(d/dt)|u(t)|^2<0$ provided $|u(t)|>|f|/\nu\lambda_1$. Similarly, (2.8) implies

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\nu|Au|^2=(f,Au), \tag{2.13}$$

which gives

$$\frac{d}{dt}\|u\|^2 + \nu\lambda_1\|u\|^2 \leq \frac{|f|^2}{\nu}$$

and thus

$$\|u(t)\|^2 \leq \|u(0)\|^2 e^{-\nu\lambda_1 t} + \frac{|f|^2}{\nu^2\lambda_1}(1 - e^{-\nu\lambda_1 t}), \qquad t \geq 0. \tag{2.14}$$

The NSE have a global attractor

$$\begin{aligned} \mathcal{A} &= \left\{u_0 \in \bigcap_{t\geq 0} S(t)H : \limsup_{t\to-\infty} |S(t)u_0| < \infty\right\} \\ &= \left\{u_0 \in \bigcap_{t\geq 0} S(t)H : |S(t)u_0| \leq \frac{|f|}{\nu\lambda_1}, t \in \mathbb{R}\right\} \\ &= \left\{u_0 \in \bigcap_{t\geq 0} S(t)H : \|S(t)u_0\| \leq \frac{|f|}{\nu\lambda_1^{1/2}}, t \in \mathbb{R}\right\}, \end{aligned} \tag{2.15}$$

which is the smallest compact subset of H which attracts all the solutions. Global attractors have been studied extensively in [BV], [CF], [H], and [T2]. The following properties will be needed:

(i) $\mathcal{A}$ is a nonempty, compact, connected subset of H;

(ii) $S(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$;

(iii) $d_F(\mathcal{A}) < \infty$, where d_F denotes the fractal dimension ([CFT], [FT]).

See [CF] or [T1] for detailed treatments of the NSE.

3. Density of trajectories of global solutions

Every solution $u(t) = S(t)u_0$ of (2.1) defined for all $t \in \mathbb{R}$ is called a *global solution*. Clearly, $u_0 \in H$ belongs to a trajectory of a global solution if and only if $u_0 \in \bigcap_{t\geq 0} S(t)H = \mathcal{G}$. Also, $\mathcal{A} \subseteq \mathcal{G}$, which shows that the union of all trajectories of global solutions is nonempty.

It is illustrative to consider again the equation

$$\dot{u} + \nu Au = f$$

briefly discussed in the introduction. For every $u_0 \in H$ there exists a solution $u(t) = S^{\text{lin}}(t)u_0 = e^{-\nu tA}\big(u_0 - (1/\nu)A^{-1}f\big) + (1/\nu)A^{-1}f$ such that $u(0) = u_0$. This solution is global if and only if

$$u_0 \in \mathcal{G}^{\text{lin}} = \left\{u_0 \in H : u_0 - \frac{1}{\nu}A^{-1}f \in \bigcap_{\alpha>0} D(e^{\alpha A})\right\},$$

where $D(e^{\alpha A})$ is the domain of definition of the operator $e^{\alpha A}$. Note that $\mathcal{G}^{\text{lin}}$ is dense in H.

The following theorem, which treats the NSE case, is the main result of this section.

THEOREM 3.1. – *The set $\mathcal{G}$, which is the set of all $u_0 \in H$ for which $S(t)u_0$ is a global solution, is dense in V'.*

In [BT] Bardos and Tartar conjectured that, for every $t_0 > 0$, $S(t_0)H$ is dense in H; Theorem 3.1 shows that $S(t_0)H$ (and even $\mathcal{G} = \bigcap_{t\geq 0} S(t)H$) is dense in H equipped with V' topology.

The main objects of study in this paper and in particular in the proof of Theorem 3.1, are the sets

$$\mathcal{M}_n = \mathcal{A}\bigcup\left\{u_0 \in \mathcal{G}\backslash\mathcal{A} : \limsup_{t\to-\infty} \frac{\|S(t)u_0\|^2}{|S(t)u_0|^2} \leq \frac{\lambda_n + \lambda_{n+1}}{2}\right\}.$$

Clearly, we have

$$S(t)\mathcal{M}_n = \mathcal{M}_n\ , \qquad t \geq 0\ , \quad n \in \mathbb{N}$$

and

$$\mathcal{A} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots.$$

If $\mathcal{M}_n^{\mathrm{lin}}$ are the analogous sets corresponding to the linear case, then

$$\mathcal{M}_n^{\mathrm{lin}} = \left\{u_0 \in H : u_0 - \frac{1}{\nu}A^{-1}f \in P_n H\right\},$$

where P_n is the orthogonal projector on the spectral space of A corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$. Note that $\bigcup_{n=1}^{\infty} \mathcal{M}_n^{\mathrm{lin}}$ is dense in H.

Regarding the NSE case, we will show that $\bigcup_{n=1}^{\infty} \mathcal{M}_n$ is dense in H with the topology of V'. The norm in V' is $|A^{-1/2}\cdot|$.

Theorem 3.1 will be proven after we establish a series of auxiliary results.

LEMMA 3.2. – *Let $\alpha \in (0,1)$ and $T > 0$ be arbitrary, and let u be a solution of the NSE such that*

$$|u(t)| > \frac{|f|}{\nu(\lambda_{n+1} - \lambda_n)\min\{\alpha, 1-\alpha\}}\ , \qquad t \in [0, T). \tag{3.1}$$

If $u(0) \in V$ and

$$\frac{\|u(0)\|^2}{|u(0)|^2} \leq \alpha\lambda_n + (1-\alpha)\lambda_{n+1} = \lambda_{n,\alpha}$$

for some n, then

$$\frac{\|u(t)\|^2}{|u(t)|^2} \leq \lambda_{n,\alpha}\ , \qquad t \in [0, T] \tag{3.2}$$

and

$$|u(t)|^2 \geq \left(|u(0)|^2 + \frac{|f|^2}{8\nu^2\lambda_{n,\alpha}^2}\right)e^{-4\nu\lambda_{n,\alpha}t} - \frac{|f|^2}{8\nu^2\lambda_{n,\alpha}^2}\ , \qquad t \in [0, T]. \tag{3.3}$$

Remark 3.3. – Lemma 3.2 and Section 2 obviously imply the following facts:

(a) If $u_0 \in \mathcal{M}_n$ and $|u_0| \geq 2|f|/\nu\lambda_1$, then there exists a unique $t_0 = t_0(u_0) \geq 0$ such that $|S(t_0)u_0| = 2|f|/\nu\lambda_1$,

$$|S(t)u_0| > 2|f|/\nu\lambda_1 , \qquad t \in [0, t_0)$$

and

$$\frac{\|S(t)u_0\|^2}{|S(t)u_0|^2} \leq \frac{\lambda_n + \lambda_{n+1}}{2} , \qquad t \in [0, t_0].$$

(b) If $u_0 \in \mathcal{M}_n \backslash \mathcal{A}$ and

$$|S(t)u_0| > \frac{2|f|}{\nu(\lambda_{n+1} - \lambda_n)} , \qquad t < T$$

for some $T \in \mathbb{R}$, then

$$\frac{\|S(t)u_0\|^2}{|S(t)u_0|^2} \leq \frac{\lambda_n + \lambda_{n+1}}{2} , \qquad t \leq T.$$

□

Proof of Lemma 3.2. – Assume first $u(0) \in D(A)$, and let $v(t) = u(t)/|u(t)|$. After some calculations, we obtain from (2.9) and (2.13)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|(A - \|v\|^2)v|^2 &= \left(\frac{f}{|u|}, (A - \|v\|^2)v\right) \\ &\leq \frac{|f|^2}{2\nu|u|^2} + \frac{\nu}{2}\big|(A - \|v\|^2)v\big|^2. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|v\|^2 + \nu\big|(A - \|v\|^2)v\big|^2 \leq \frac{|f|^2}{\nu|u|^2} , \qquad t \in [0, T]. \tag{3.4}$$

Since

$$|(A - \lambda)v| \geq \Big(\min_{n\in\mathbb{N}} |\lambda - \lambda_n|\Big)|v| , \qquad \lambda \in \mathbb{R} , \quad v \in D(A)$$

we get

$$\frac{d}{dt}\|v\|^2 + \nu\Big(\min_{n\in\mathbb{N}}\big| \|v\|^2 - \lambda_n\big|\Big)^2 \leq \frac{|f|^2}{\nu|u|^2} , \qquad t \in [0, T]. \tag{3.5}$$

If $\|v(t_0)\|^2 = \lambda_{n,\alpha}$, for some $t_0 \in [0, T)$, then

$$\frac{d}{dt}\|v\|^2\Big|_{t=t_0} \leq \frac{|f|^2}{\nu|u(t_0)|^2} - \nu(\lambda_{n+1} - \lambda_n)^2(\min\{\alpha, 1 - \alpha\})^2$$

by (3.5). The right hand side is negative by (3.1), so (3.2) follows.

In order to prove (3.3), we use (2.9) and (3.2):

$$\frac{d}{dt}|u|^2 = -2\nu\|u\|^2 + 2(f, u) \geq -4\nu\lambda_{n,\alpha}|u|^2 - \frac{|f|^2}{2\nu\lambda_{n,\alpha}} \tag{3.6}$$

and (3.3) follows.

Now, assume $u(0) \in V$, and fix $\epsilon \in (0, T/2)$. There exist $\epsilon', \epsilon'' \in (0, \epsilon)$ such that $\alpha - \epsilon'' > 0$, $u(\epsilon') \in D(A)$, $\|v(\epsilon')\|^2 \leq \lambda_{n,\alpha-\epsilon''}$, and

$$|u(t)| > \frac{|f|}{\nu\lambda_1 \min\{\alpha - \epsilon'', 1 - (\alpha - \epsilon'')\}}, \qquad t \in [\epsilon', T - \epsilon'].$$

Then $\|v(t)\|^2 \leq \lambda_{n,\alpha-\epsilon''}$ for $t \in [\epsilon', T - \epsilon']$ by the first part of the proof. As we let $\epsilon \to 0$, we get (3.2), while (3.3) directly follows from (3.2). □

We recall that P_n is the orthogonal projector in H on the spectral space of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; let also $Q_n = I - P_n$.

LEMMA 3.4. – *If*

$$\|u_0\|^2 \leq \frac{\lambda_n + \lambda_{n+1}}{2}|u_0|^2$$

for some $u_0 \in V$ *and* $n \in \mathbb{N}$, *then* $|Q_n u_0|^2 \leq \gamma_n |P_n u_0|^2$ *where* $\gamma_n = (\lambda_{n+1} + \lambda_n)/(\lambda_{n+1} - \lambda_n)$, *and*

$$|u_0|^2 \leq \frac{2\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}|P_n u_0|^2.$$

Proof of Lemma 3.4. – We have

$$|Q_n u_0|^2 \leq \frac{1}{\lambda_{n+1}}\|Q_n u_0\|^2 \leq \frac{1}{\lambda_{n+1}}\|u_0\|^2 \leq \frac{\lambda_n + \lambda_{n+1}}{2\lambda_{n+1}}|P_n u_0|^2 + \frac{\lambda_n + \lambda_{n+1}}{2\lambda_{n+1}}|Q_n u_0|^2$$

and both assertions follow. □

LEMMA 3.5. – *Let* $p_0 \in P_n H$ *for some* n. *Then, for every* $t_0 > 0$, *there exists* $v_0 \in P_n H$ *such that* $P_n S(t_0) v_0 = p_0$.

For the proof we will need, in addition to the previous two lemmas, the following well-known fact:

THEOREM 3.6 (Brouwer). – *Let* $B(r) \subseteq \mathbb{R}^n$ *be a closed ball with center 0 and radius* r. *If* $g: B(r) \to B(r)$ *is a continuous mapping, and if* $g(x) = x$ *for all* $x \in \partial B(r)$, *then* g *is onto.*

Proof of Lemma 3.5. – Let $\Gamma_n(r) = \{u_0 \in P_n H : |u_0| \geq r\}$. We will first show that there exists

$$r_0 \geq \frac{|f|}{\nu\lambda_1} \tag{3.7}$$

such that

$$|P_n S(t) u_0| > |p_0|, \qquad u_0 \in \Gamma_n(r_0), \quad t \in [0, t_0]. \tag{3.8}$$

It is sufficient only to consider the case $|p_0| \geq 2|f|/\nu\lambda_1$. Choose any $u_0 \in P_n H$ such that $|u_0| > |p_0|$. It follows from (2.10) that there exists

$$\tau(u_0) = \min\left\{\tau_0 > 0 : |P_n S(\tau_0) u_0| = \frac{2|f|}{\nu\lambda_1}\right\} > 0.$$

Note that $\|u_0\|/|u_0|^2 \leq (\lambda_n + \lambda_{n+1})/2$. Hence, by Lemma 3.2 (with $\alpha = 1/2$),

$$\|S(t)u_0\|^2 \leq \frac{\lambda_n + \lambda_{n+1}}{2}|S(t)u_0|^2 , \qquad t \in [0, \tau(u_0)]$$

and

$$|S(t)u_0|^2 \geq |u_0|^2 e^{-4\nu\lambda_{n+1}t} - \frac{|f|^2}{8\nu^2\lambda_n^2} , \qquad t \in [0, \tau(u_0)].$$

Lemma 3.4 now implies $|Q_n S(t)u_0|^2 \leq \gamma_n |P_n S(t)u_0|^2$, for $t \in [0, \tau(u_0)]$, whence

$$\begin{aligned} |P_n S(t)u_0|^2 &\geq \frac{1}{\gamma_n + 1}|S(t)u_0|^2 \\ &\geq \frac{1}{\gamma_n + 1}\left(|u_0|^2 e^{-4\nu\lambda_{n+1}t} - \frac{|f|^2}{8\nu^2\lambda_n^2}\right) , \qquad t \in [0, \tau(u_0)]. \end{aligned} \tag{3.9}$$

We fix any r_0 which satisfies (3.7) and

$$\frac{1}{\gamma_n + 1}\left(r_0^2 e^{-4\nu\lambda_{n+1}t_0} - \frac{|f|^2}{8\nu^2\lambda_n^2}\right) > |p_0|^2. \tag{3.10}$$

Let $|u_0| \geq r_0$, and set $t = \tau(u_0)$ in (3.9):

$$|p_0|^2 \geq \frac{4|f|^2}{\nu^2\lambda_1^2} = |P_n S(\tau_0)u_0|^2 \geq \frac{1}{\gamma_n + 1}\left(r_0^2 e^{-4\nu\lambda_{n+1}\tau(u_0)} - \frac{|f|^2}{8\nu^2\lambda_n^2}\right).$$

This and (3.10) imply $t_0 < \tau(u_0)$. Therefore, (3.9) holds for $t \in [0, t_0]$, and (3.8) follows from (3.9) and (3.10).

In order to apply Theorem 3.6, we will find a suitable $g: P_n H \to P_n H$. First, we choose a continuous function $\theta: \mathbb{R} \to \mathbb{R}$ such that $\theta(x) = 1$ for $x \leq r_0$, and $\theta(x) = 0$ for $x \geq 2r_0$. Define

$$g(u_0) = P_n S\big(\theta(|u_0|)t_0\big)u_0 , \qquad u_0 \in P_n H.$$

Note that $g(B^H(2r_0) \cap P_n H) \subseteq B^H(2r_0) \cap P_n H$ (where $B^H(r) = \{u_0 \in H : |u_0| \leq r\}$) by (3.7) and by

$$S(t)B^H(r) \subseteq B^H(r) , \qquad r \geq \frac{|f|}{\nu\lambda_1} , \quad t \geq 0.$$

Also, g is continuous, and it satisfies $g(u_0) = u_0$ if $|u_0| = 2r_0$. By Theorem 3.6, there exists $v_0 \in P_n H$ such that $g(v_0) = p_0$. Now, because of (3.8), we have $|v_0| \leq r_0$; thus, $g(v_0) = P_n S(t_0)u_0 = p_0$. □

In the next lemma we provide the main ingredient for the construction of global solutions:

LEMMA 3.7. – *Let $u_1, u_2, \ldots \in H$, and let $t_1 > t_2 > t_3 > \cdots$ be such that $\lim_{n\to\infty} t_n = -\infty$. Suppose that $S(t)u_j$ is a solution for $t \in [t_j, \infty)$. Then the following two statements hold:*

(a) If there is $\tilde{t} \in \mathbb{R}$ *such that* $\limsup_{k\to\infty} \|S(t)u_k\| < \infty$ *for all* $t \le \tilde{t}$, *then there exists* $u_\infty \in \mathcal{G}$ *and a subsequence* $\{u_{k_j}\}_{j=1}^\infty$ *of* $\{u_k\}_{k=1}^\infty$ *such that*

$$\lim_{j\to\infty} |S(t)u_{k_j} - S(t)u_\infty| = 0 , \qquad t \in \mathbb{R}. \tag{3.11}$$

(b) If

$$|u_k| \le M , \qquad k \in \mathbb{N} \tag{3.12}$$

for some constant M, *and if*

$$|S(t_k)u_k| \ge 2|f|/\nu\lambda_1 , \qquad k \in \mathbb{N} \tag{3.13}$$

with

$$\|S(t_k)u_k\|^2 \le \frac{\lambda_n + \lambda_{n+1}}{2} |S(t_k)u_k|^2 , \qquad k \in \mathbb{N} \tag{3.14}$$

for some fixed n, *then there exists* $u_\infty \in \mathcal{M}_n$ *and a subsequence* $\{u_{k_j}\}_{j=1}^\infty$ *of* $\{u_k\}_{k=1}^\infty$ *such that (3.11) holds.*

In the proof we will need the following elementary facts:

Remark 3.8. – If $\{u_k\}_{k=1}^\infty$ is a bounded sequence in V, and if $\lim_{k\to\infty} |u_k - u_0| = 0$ for some $u_0 \in H$, then $u_0 \in V$ and $\|u_0\| \le \liminf_{k\to\infty} \|u_k\|$. Likewise, if $\{u_k\}_{k=1}^\infty$ is a bounded sequence in V, and if $\lim_{k\to\infty} |A^{-1/2}(u_k - u_0)| = 0$ for some $u_0 \in V'$, then $u_0 \in V$ and $\|u_0\| \le \liminf_{k\to\infty} \|u_k\|$. □

Proof of Lemma 3.7. – (a) Since the imbedding $i_1 : V \to H$ is compact, we may use the Cantor diagonal process to find a subsequence $\{u_{k_j}\}_{j=1}^\infty$ of $\{u_k\}_{k=1}^\infty$ such that the limits

$$\lim_{j\to\infty} S(t_{k_i})u_{k_j} = v_i \in H , \qquad i \in \mathbb{N},$$

(the limits being taken in H) exist, and such that $t_{k_j} \le \tilde{t}$ for $j \in \mathbb{N}$. Since $S(t): H \to H$ is continuous for every $t \ge 0$, we get

$$v_i = S(t_{k_i} - t_{k_j})v_j , \qquad j \le i , \quad i, j \in \mathbb{N}.$$

Letting $u_\infty = S(-t_{k_1})v_1$, we obtain

$$S(-t_{k_j})v_j = u_\infty , \qquad j \in \mathbb{N}.$$

Finally, (a) follows from continuity of S.

(b) Without loss of generality, we may assume $M > |f|/\nu\lambda_1$. First, fix $k \in \mathbb{N}$. As in Remark 3.3(a), there exists a unique $\alpha_k \ge t_k$ such that $|S(\alpha_k)u_k| = 2|f|/\nu\lambda_1$,

$$|S(t)u_k| \le \frac{2|f|}{\nu\lambda_1} , \qquad t \ge \alpha_k \tag{3.15}$$

and

$$\|S(t)u_k\|^2 \le \frac{\lambda_n + \lambda_{n+1}}{2} |S(t)u_k|^2 , \qquad t \in [t_k, \alpha_k]. \tag{3.16}$$

Moreover, (3.6), (3.13), and (3.14) imply

$$(3.17)\qquad \begin{aligned} |S(t)u_k|^2 &\le \left(|S(\alpha_k)u_k|^2 + \frac{|f|^2}{8\nu^2\lambda_1^2}\right)e^{-4\nu\lambda_{n+1}(t-\alpha_k)} \\ &= \left(\frac{4|f|^2}{\nu^2\lambda_1^2} + \frac{|f|^2}{8\nu^2\lambda_1^2}\right)e^{-4\nu\lambda_{n+1}(t-\alpha_k)} \\ &= C_1 e^{-4\nu\lambda_{n+1}(t-\alpha_k)}, \quad t \in [t_k, \alpha_k], \end{aligned}$$

where C_1, C_2, ... are various constants. On the other hand, (2.14) and (3.16) give

$$(3.18)\qquad \|S(t)u_k\|^2 \le \max\left\{\frac{\lambda_n + \lambda_{n+1}}{2}\frac{4|f|^2}{\nu^2\lambda_1^2}, \frac{|f|^2}{\nu^2\lambda_1}\right\} = C_2\ , \qquad t \ge \alpha_k.$$

Note that an upper bound on α_k is

$$\alpha_k \le \frac{1}{\nu\lambda_1}\log\frac{\nu^2\lambda_1^2 M^2 - |f|^2}{3|f|^2} = t_M$$

which can be obtained from (3.12), (3.15), and (2.11). By this estimate, together with (3.16) and (3.17), we get

$$\begin{aligned} \|S(t)u_k\|^2 &\le \frac{\lambda_n + \lambda_{n+1}}{2} C_1 e^{-4\nu\lambda_{n+1}(t-\alpha_k)} \\ &\le C_3 e^{-4\nu\lambda_{n+1}(t-t_M)}\ , \qquad t \in [t_k, \alpha_k]. \end{aligned}$$

This and (3.18) yield

$$\|S(t)u_k\|^2 \le \max\{C_2, C_3 e^{-4\nu\lambda_{n+1}(t-t_M)}\}\ , \qquad t \ge t_k.$$

So, the assumptions of Lemma 3.7(a) are satisfied. Hence, we get (3.11) for $u_\infty \in \mathcal{G}$ and a suitable subsequence $\{u_{k_j}\}_{j=1}^\infty$ of $\{u_k\}_{k=1}^\infty$. If $\liminf_{k\to\infty}\alpha_k = -\infty$, then (3.11) and (3.15) imply $|S(t)u_\infty| \le 2|f|/\nu\lambda_1$ for $t \in \mathbb{R}$, and thus $u_\infty \in \mathcal{A}$. If on the other hand $\liminf_{k\to\infty}\alpha_k = \alpha_\infty > -\infty$, Remark 3.8, together with (3.11) and (3.16), gives

$$\|S(t)u_\infty\|^2 \le \frac{\lambda_n + \lambda_{n+1}}{2}|S(t)u_\infty|^2\ , \qquad t \le \alpha_\infty$$

and thus again $u_\infty \in \mathcal{M}_n$. □

Theorem 3.1 will be an easy consequence of the following lemma:

LEMMA 3.9. – *If $p_0 \in P_nH$, for some n, then there exists a global solution $S(t)u_\infty$ such that:*

(a) $u_\infty \in \mathcal{M}_n$;

(b) $P_n u_\infty = p_0$;

(c) $|Q_n u_\infty| \le \max\{2|f|/\nu\lambda_1, \gamma_n^{1/2}|p_0|\}$ *where* $\gamma_n = (\lambda_{n+1} + \lambda_n)/(\lambda_{n+1} - \lambda_n)$.

Before the proof, we will show that Theorem 3.1 is a direct consequence of Lemma 3.9:

Proof of Theorem 3.1. – Let $u_0 \in H$ be arbitrary. For any $n \in \mathbb{N}$, Lemma 3.9 provides $u_n \in H$ for which $S(t)u_n$ is a global solution, $P_n u_n = P_n u_0$, and

$$|Q_n u_n| \le \max\left\{\frac{2|f|}{\nu\lambda_1}, \gamma_n^{1/2}|P_n u_0|\right\}.$$

All these facts imply

$$\begin{aligned}
|A^{-1/2}(u_n - u_0)| &= |A^{-1/2}Q_n(u_n - u_0)| \\
&\le \lambda_{n+1}^{-1/2}|Q_n(u_n - u_0)| \\
&\le \lambda_{n+1}^{-1/2}\left(|Q_n u_0| + \max\left\{\frac{2|f|}{\nu\lambda_1}, \left(\frac{\lambda_{n+1}+\lambda_n}{\lambda_{n+1}-\lambda_n}\right)^{1/2}|P_n u_0|\right\}\right) \\
&\le \lambda_{n+1}^{-1/2}|u_0| + \max\left\{\frac{2|f|}{\nu\lambda_1\lambda_{n+1}^{1/2}}, \frac{\sqrt{2}}{(\lambda_{n+1}-\lambda_n)^{1/2}}|u_0|\right\}
\end{aligned}$$

for all n. By virtue of (2.3) and (2.4), we obtain $\liminf_{n\to\infty} |A^{-1/2}(u_n - u_0)| = 0$, and the theorem is proven. □

Proof of Lemma 3.9. – First, note that (c) follows from (a) and (b): If $|u_\infty| \le 2|f|/\nu\lambda_1$, then $|Q_n u_\infty| \le |u_\infty| \le 2|f|/\nu\lambda_1$. If on the other hand $|u_\infty| \ge 2|f|/\nu\lambda_1$, we get

$$\|u_\infty\|^2 \le \frac{\lambda_n + \lambda_{n+1}}{2}|u_\infty|^2$$

by (a) and Remark 3.3(a). Lemma 3.4 and (b) then complete (c).

It remains to establish (a) and (b): Choose a sequence $0 > t_1 > t_2 > \cdots$ such that $\lim_{k\to\infty} t_k = -\infty$. By Lemma 3.5, there exist $u_1, u_2, \ldots \in H$ and $p_1, p_2, \ldots \in P_n H$ such that

$$S(-t_k)p_k = u_k\ , \qquad k \in \mathbb{N}$$

and

$$P_n u_k = p_0\ , \qquad k \in \mathbb{N}. \tag{3.19}$$

Consider the following sequence of solutions:

$$U_k(t) = S(t)u_k\ , \qquad t \ge t_k.$$

We consider two cases:

Case 1: $|p_k| \le 2|f|/\nu\lambda_1$ for infinitely many $k \in \mathbb{N}$.

By passing to a subsequence, we may assume

$$|p_k| \le \frac{2|f|}{\nu\lambda_1}\ , \qquad k \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$, and note

$$\|U_k(t_k)\| = \|p_k\| \le \lambda_n^{1/2}|p_k| \le \frac{2\lambda_n^{1/2}|f|}{\nu\lambda_1}.$$

The inequality (2.14) then gives $\|U_k(t)\| \le 2\lambda_n^{1/2}|f|/\nu\lambda_1$ for $t \ge t_k$ whence

$$(3.20) \qquad |U_k(t)| \le \lambda_1^{-1/2}\|U_k(t)\| \le \frac{2\lambda_n^{1/2}|f|}{\nu\lambda_1^{3/2}}\,, \qquad t \ge t_k.$$

According to Lemma 3.7, we may, by passing to a subsequence, assume that

$$(3.21) \qquad \lim_{k\to\infty} |U_k(t) - S(t)u_\infty| = 0\,, \qquad t \in \mathbb{R}$$

for some $u_\infty \in \mathcal{G}$. Now, (3.19) and (3.21) imply (b), while (3.20) and (3.21) show that $|S(t)u_\infty| \le 2\lambda_n^{1/2}|f|/\nu\lambda_1^{3/2}$ for $t \in \mathbb{R}$. Because of (2.15) we get $u_\infty \in \mathcal{A}$ and thus (a).

Case 2: $|p_k| \ge 2|f|/\nu\lambda_1$ for infinitely many $k \in \mathbb{N}$.

By passing to a subsequence, we may assume

$$|U_k(t_k)| = |p_k| \ge \frac{2|f|}{\nu\lambda_1}\,, \qquad k \in \mathbb{N}.$$

Note that, for each $k \in \mathbb{N}$, either $|u_k| \le 2|f|/\nu\lambda_1$ or $|u_k| > 2|f|/\nu\lambda_1$ in which case $\|u_k\|^2/|u_k|^2 \le (\lambda_n + \lambda_{n+1})/2$ by Lemma 3.2. Thus,

$$|u_k| \le \max\left\{\frac{2|f|}{\nu\lambda_1}, \left(\frac{2\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right)^{1/2} |p_0|\right\}\,, \qquad k \in \mathbb{N}$$

due to Lemma 3.4. Since also

$$\frac{\|U_k(t_k)\|^2}{|U_k(t_k)|^2} = \frac{\|p_k\|^2}{|p_k|^2} \le \lambda_n\,, \qquad k \in \mathbb{N}$$

the assumptions of Lemma 3.7(b) are fulfilled. By passing to a subsequence, we obtain $u_\infty \in \mathcal{M}_n$ such that (3.21) holds. So, (a) is valid, and (b) follows from (3.19) and (3.21).□

Remark 3.10. – At this point, we are unable to prove that $\mathcal{G}$, which is the set of initial data which lead to a global solution, is actually dense in the norm $|\cdot|$. We can however show that we have $0 \in \bar{\mathcal{G}}$, where the closure is taken in H.

Let $\epsilon > 0$ be arbitrary. Choose n so that

$$\frac{2|f|}{\nu(\lambda_{n+1} - \lambda_n)} \le \epsilon$$

and

$$(3.22) \qquad \dim P_n H > d_F(\mathcal{A}).$$

We claim that there exists $u_0 \in \mathcal{M}_n \backslash \mathcal{A}$ such that $|u_0| \le \epsilon$. Indeed, suppose that this is not true. Then, by Remark 3.3(b),

$$(3.23) \qquad \frac{\|u_0\|^2}{|u_0|^2} \le \frac{\lambda_n + \lambda_{n+1}}{2}\,, \qquad u_0 \in \mathcal{M}_n \backslash \mathcal{A}$$

and

$$|u_0| > \epsilon\,, \qquad u_0 \in \mathcal{M}_n \backslash \mathcal{A}. \tag{3.24}$$

Consider the set

$$S = \left\{ p \in P_n H : |p|^2 \leq \frac{(\lambda_{n+1} - \lambda_n)\epsilon^2}{2\lambda_{n+1}} \right\}.$$

Note that $P_n \mathcal{M}_n \supseteq S$ and $P_n(\mathcal{M}_n \backslash \mathcal{A}) \cap S = \emptyset$: The first fact follows from Lemma 3.9, while if $u_0 \in \mathcal{M}_n \backslash \mathcal{A}$, we have by Lemma 3.4, (3.23), and (3.24)

$$|P_n u_0|^2 \geq \frac{\lambda_{n+1} - \lambda_n}{2\lambda_{n+1}} |u_0|^2 > \frac{(\lambda_{n+1} - \lambda_n)\epsilon^2}{2\lambda_{n+1}}.$$

Hence, $P_n \mathcal{A} \supseteq S$, and $d_F(S) = \dim P_n H$ contradicts (3.22). □

4. Further properties of the invariant sets $\mathcal{M}_n$

For $n = 1, 2, \ldots$, let

$$\mathcal{C}_n = \left\{ x \in V : \|x\|^2 \leq \frac{\lambda_n + \lambda_{n+1}}{2} |x|^2 \right\}.$$

Also, recall that $B^H(r) = \{u_0 \in H : |u_0| \leq r\}$.

THEOREM 4.1. – *For each $n \in \mathbb{N}$, $\mathcal{M}_n$ is a connected, locally compact subset of H, and*

$$P_n \mathcal{M}_n = P_n H\,; \tag{4.1}$$

moreover,

$$\mathcal{M}_n \subseteq B^H\left(\frac{2|f|}{\nu\lambda_1}\right) \bigcup \mathcal{C}_n. \tag{4.2}$$

Proof. – The equality (4.1) is already contained in Lemma 3.9, while (4.2) follows from Remark 3.3(a), (2.11), and (2.15). It is also easy to check that every S-invariant set which includes $\mathcal{A}$ is connected.

Since $\mathcal{A}$ is compact (in H), it only remains to check that every sequence $u_1, u_2, \ldots \in \mathcal{M}_n \backslash \mathcal{A}$ such that

$$\sup_{k\in\mathbb{N}} |u_k| \leq M < \infty$$

for some M, has a subsequence converging to an element in $\mathcal{M}_n$. Due to $\lim_{t\to-\infty} |S(t)u_k| = \infty$ for $k \in \mathbb{N}$, we can find a sequence $t_1 > t_2 > t_3 > \cdots$ such that $\lim_{k\to\infty} t_k = -\infty$ and $\inf_{k\in\mathbb{N}} |S(t_k)u_k| \geq 2|f|/\nu\lambda_1$. An application of Lemma 3.7 concludes the proof since (3.14) follows from Remark 3.3(a). □

THEOREM 4.2. – *If $u(t) = S(t)u_0$ $(t \in \mathbb{R})$ is a global solution* (i.e., $u_0 \in \mathcal{G}$), *and if $u_0 \notin \mathcal{A}$, then exactly one of the following two possibilities occurs:*

(a)

$$\lim_{t\to-\infty}\frac{\|u(t)\|^2}{|u(t)|^2}=\lambda_n \tag{4.3}$$

for some n, *in which case* $\lim_{t\to-\infty}(\log|u(t)|)/|t| = \nu\lambda_n$;

(b)

$$\lim_{t\to-\infty}\frac{\|u(t)\|^2}{|u(t)|^2}=\infty \tag{4.4}$$

in which case $\lim_{t\to-\infty}(\log|u(t)|)/|t| = \infty$.

This theorem readily yields the following new characterization of the invariant sets $\mathcal{M}_n$:

COROLLARY 4.3. – *For each* $n\in\mathbb{N}$, *we have*

$$\begin{aligned}\mathcal{M}_n &= \{u_0\in\mathcal{G} : |S(t)u_0| = \mathcal{O}(e^{(1+\epsilon)\nu\lambda_n|t|}) \text{ as } t\to-\infty, \forall\epsilon>0\}\\ &= \{u_0\in\mathcal{G} : |S(t)u_0| = \mathcal{O}(e^{\nu(\lambda_n+\lambda_{n+1})|t|/2}) \text{ as } t\to-\infty\}\\ &= \mathcal{A}\bigcup\left\{u_0\in\mathcal{G}\backslash\mathcal{A} : \lim_{t\to-\infty}\frac{\|S(t)u_0\|^2}{|S(t)u_0|^2}\in\{\lambda_1,\lambda_2,\dots,\lambda_n\}\right\}\\ &= \mathcal{A}\bigcup\left\{u_0\in\mathcal{G}\backslash\mathcal{A} : \lim_{t\to-\infty}\frac{\log|S(t)u_0|}{|t|}\in\{\nu\lambda_1,\nu\lambda_2,\dots,\nu\lambda_n\}\right\}.\end{aligned}$$

Proof of Theorem 4.2. – For simplicity, we introduce $v(t) = u(t)/|u(t)|$. Lemma 3.2 shows that, for any $a\geq 0$, $\liminf_{t\to-\infty}\|v(t)\|\leq a$ implies $\limsup_{t\to-\infty}\|v(t)\|\leq a$. Therefore, $\lim_{t\to-\infty}\|v(t)\|\in[0,\infty]$ exists.

Suppose that this limit is finite. Note that $\lim_{t\to-\infty}|u(t)|=\infty$ since $u_0\notin\mathcal{A}$; therefore, (3.5) implies

$$\begin{aligned}0\leq \limsup_{t\to-\infty}\frac{d}{dt}\|v\|^2 &\leq \limsup_{t\to-\infty}\frac{|f|^2}{\nu|u|^2} - \nu\liminf_{t\to-\infty}\left(\min_{n\in\mathbb{N}}\left|\,\|v\|^2-\lambda_n\right|\right)^2\\ &= -\nu\left(\min_{n\in\mathbb{N}}\left|\lim_{t\to-\infty}\|v\|^2-\lambda_n\right|\right)^2.\end{aligned}$$

This gives (4.3) for some n.

Assume (4.3), and fix $\epsilon>0$. There exists $t_0\in\mathbb{R}$ such that

$$(1-\epsilon)\lambda_n\leq\|v(t)\|^2\leq(1+\epsilon)\lambda_n\ , \qquad t\leq t_0.$$

Therefore, by (2.9),

$$\begin{aligned}\frac{d}{dt}|u|^2 &= -2\nu\|v\|^2|u|^2+2(f,u)\\ &\leq -2(1-\epsilon)\nu\lambda_n|u|^2+2|f|\,|u|\\ &\leq -2(1-\epsilon)\nu\lambda_n|u|^2+\epsilon\nu\lambda_n|u|^2+\frac{|f|^2}{\epsilon\nu\lambda_n}\\ &= -(2-3\epsilon)\nu\lambda_n|u|^2+\frac{|f|^2}{\epsilon\nu\lambda_n}\ , \qquad t\leq t_0\end{aligned} \tag{4.5}$$

and similarly

$$\frac{d}{dt}|u|^2 \geq -2(1+\epsilon)\nu\lambda_n|u|^2 - 2|f|\,|u| \tag{4.6}$$
$$\geq -2(1+\epsilon)\nu\lambda_n|u|^2 - \epsilon\nu\lambda_n|u|^2 - \frac{|f|^2}{\epsilon\nu\lambda_n}$$
$$= -(2+3\epsilon)\nu\lambda_n|u|^2 - \frac{|f|^2}{\epsilon\nu\lambda_n}, \qquad t \leq t_0.$$

From (4.5) and (4.6) it follows

$$\left(1-\frac{3\epsilon}{2}\right)\nu\lambda_n \leq \liminf_{t\to-\infty}\frac{\log|u(t)|}{|t|} \leq \limsup_{t\to-\infty}\frac{\log|u(t)|}{|t|} \leq \left(1+\frac{3\epsilon}{2}\right)\nu\lambda_n.$$

Letting $\epsilon \to 0$, we get $\lim_{t\to-\infty}(\log|u(t)|)/|t| = \nu\lambda_n$.

Now, assume (4.4). For each $n \in \mathbb{N}$ and $\epsilon > 0$, there exists $t_0 \in \mathbb{R}$ such that

$$\|v(t)\|^2 \geq (1-\epsilon)\lambda_n, \qquad t \leq t_0.$$

As above,

$$\liminf_{t\to-\infty}\frac{\log|u(t)|}{|t|} \geq \left(1-\frac{3\epsilon}{2}\right)\nu\lambda_n.$$

Since $n \in \mathbb{N}$ and $\epsilon > 0$ were arbitrary, we obtain $\lim_{t\to-\infty}(\log|u(t)|)/|t| = \infty$. □

If u is a global solution such that $u(0) \notin \mathcal{A}$ and $\lim_{t\to-\infty}\|u(t)\|^2/|u(t)|^2 < \infty$, then Theorem 4.2 guarantees that $\|u(t)\|^2/|u(t)|^2$ and $\log|u(t)|/|t|$ converge to λ_n and $\nu\lambda_n$ respectively, for some n, as $t \to -\infty$. The next theorem estimates the convergence rates of both quantities.

THEOREM 4.4. – *If u is a global solution such that $u(0) \notin \mathcal{A}$, and if $\lim_{t\to-\infty}\|u(t)\|^2/|u(t)|^2 = \lambda_n$ for some n, then the following statements hold:*

(a) We have

$$\limsup_{t\to-\infty}\left(\frac{\|u(t)\|^2}{|u(t)|^2} - \lambda_n\right)e^{-\nu\lambda_1 t} < \infty \tag{4.7}$$

and

$$\liminf_{t\to-\infty}\left(\frac{\|u(t)\|^2}{|u(t)|^2} - \lambda_n\right)|t| > -\infty. \tag{4.8}$$

(b) For every $\mu > 1$,

$$\limsup_{t\to-\infty}|u(t)|e^{\nu\lambda_n t} < \infty \tag{4.9}$$

and

$$\liminf_{t\to-\infty}|u(t)|\,|t|^{\mu}e^{\nu\lambda_n t} > 0. \tag{4.10}$$

Theorem 4.4 clearly implies the following improvement of the first characterization of $\mathcal{M}_n$ given in Corollary 4.3:

COROLLARY 4.5. – *For each $n \in \mathbb{N}$, we have*

$$\mathcal{M}_n = \{u_0 \in \mathcal{G} : |S(t)u_0| = \mathcal{O}(e^{\nu\lambda_n|t|}) \text{ as } t \to -\infty\}.$$

Proof of Theorem 4.4. – Denote $v = u/|u|$, and fix $\epsilon \in (0, 1/2)$.

(a) Choose $t_0 \in \mathbb{R}$ so that $|u(t_0)| = 2|f|/\nu\lambda_1$. Then (2.12) implies

$$|u(t)|^2 \geq C_4 e^{-\nu\lambda_1 t}, \qquad t \leq t_0, \tag{4.11}$$

where $C_4 = (3|f|^2/\nu^2\lambda_1^2)e^{\nu\lambda_1 t_0}$. Integrating (3.5) between $-\infty$ and t and using (4.11), we obtain

$$\|v(t)\|^2 - \lambda_n \leq \int_{-\infty}^{t} \frac{|f|^2}{\nu|u(\tau)|^2}\, d\tau \leq C_5 e^{\nu\lambda_1 t}, \qquad t \leq t_0 \tag{4.12}$$

since $\lim_{t\to-\infty} \|v(t)\|^2 = \lambda_n$. This gives (4.7).

Now, choose $t_0 \in \mathbb{R}$ so that

$$(\lambda_{n-1} + \lambda_n)/2 \leq \|v(t)\|^2 \leq (\lambda_n + \lambda_{n+1})/2, \qquad t \leq t_0$$

and (4.11) hold. Then $y = \|v\|^2 - \lambda_n$ satisfies

$$\dot{y}(t) + \nu y^2(t) \leq \frac{|f|^2}{\nu|u(t)|^2} \leq C_6 e^{\nu\lambda_1 t}, \qquad t \leq t_0. \tag{4.13}$$

We distinguish three possibilities:

Case 1: There exists $t_1 \in \mathbb{R}$ such that $y(t) \leq -(C_6/\epsilon\nu)^{1/2} e^{\nu\lambda_1 t/2}$ for $t \leq t_1$.

Then (4.13) implies

$$\dot{y}(t) + (1-\epsilon)\nu y^2(t) \leq 0 \tag{4.14}$$

for $t \leq t_1$ whence

$$y(t) \geq -\frac{1}{(1-\epsilon)\nu(t_1 - t) - y(t_1)^{-1}}, \qquad t \leq t_1$$

and (4.8) holds.

Case 2: There exists $t_1 \in \mathbb{R}$ such that $y(t) \geq -(C_6/\epsilon\nu)^{1/2} e^{\nu\lambda_1 t/2}$ for $t \leq t_1$.

In this case (4.8) follows immediately.

Case 3: None of the cases 1 or 2 occurs.

Let $t_1 < t_2 \leq t_0$ be such that

$$y(t) \leq -\left(\frac{C_6}{\epsilon\nu}\right)^{1/2} e^{\nu\lambda_1 t/2}, \qquad t \in [t_1, t_2]$$

and assume that when $t = t_1$ or $t = t_2$, we have the equality sign. Then we have (4.14) for $t \in [t_1, t_2]$, and thus

$$y(t) \geq -\frac{1}{(1-\epsilon)\nu(t_2 - t) + (\epsilon\nu/C_6)^{1/2} e^{-\nu\lambda_1 t_2/2}}, \qquad t \in [t_1, t_2].$$

Choose $t_0' < \min\{0, t_0\}$ so that

$$(1-\epsilon)\nu t + \left(\frac{\epsilon\nu}{C_6}\right)^{1/2} e^{-\nu\lambda_1 t/2} \geq 0, \qquad t \leq t_0'.$$

Then

$$y(t) \geq \frac{1}{(1-\epsilon)\nu t}, \qquad t \in [t_1, t_2]$$

provided t_1 and t_2 are chosen so that $t_2 \leq t_0'$. This proves that in Case 3

$$y(t) \geq \min\left\{\frac{1}{(1-\epsilon)\nu t}, -\left(\frac{C_6}{\epsilon\nu}\right)^{1/2} e^{\nu\lambda_1 t/2}\right\}, \qquad t \leq t_0''$$

for a suitable $t_0'' < t_0'$, and (4.8) is established.

(b) Using (2.9), we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|v\|^2|u|^2 = (f, u) \geq -|f|\,|u|$$

which because of (4.12) leads to

$$\frac{d}{dt}|u| + \nu\left(\lambda_n + C_5 e^{\nu\lambda_1 t}\right)|u| \geq -|f|, \qquad t \leq t_0. \tag{4.15}$$

Then

$$|u(t)| \leq |u(t_0)| e^{\alpha(t_0)-\alpha(t)} + |f| \int_t^{t_0} e^{\alpha(\tau)-\alpha(t)}\, d\tau, \qquad t \leq t_0$$

where

$$\alpha(t) = \nu\lambda_n t + \frac{C_5}{\lambda_1} e^{\nu\lambda_1 t}.$$

Using the estimate $|(C_5/\lambda_1)e^{\nu\lambda_1 t}| \leq C_7$ for $t \leq t_0$, where C_7 is a suitable constant, we obtain

$$|u(t)| \leq C_8 e^{-\nu\lambda_n t}, \qquad t \leq t_0$$

which concludes the proof of (4.9).

Let $\mu > 1$ be arbitrary. The proof of (4.8) shows that

$$\|v(t)\|^2 - \lambda_n \geq \frac{\mu}{\nu t}, \qquad t \leq t_0,$$

for a suitable $t_0 < 0$ (possibly different from above). Similarly as we obtained (4.15), we get

$$\frac{d}{dt}|u| + \left(\nu\lambda_n + \frac{\mu}{t}\right)|u| \le |f| , \qquad t \le t_0.$$

Therefore,

$$\begin{aligned} |u(t)|e^{\nu\lambda_n t+\mu \log|t|} &\ge |u(t_1)|e^{\nu\lambda_n t_1+\mu\log|t_1|} - |f| \int_t^{t_1} e^{\nu\lambda_n\tau+\mu\log|\tau|}\,d\tau \\ &\ge |u(t_1)|e^{\nu\lambda_n t_1+\mu\log|t_1|} - |f| \int_{-\infty}^{t_1} e^{\nu\lambda_n\tau+\mu\log|\tau|}\,d\tau, \end{aligned}$$

provided $t \le t_1 \le t_0$. We may assume that t_0 is chosen so small that we have (4.11) and $\mu \log|t| \le -\nu\lambda_1 t/4$ for $t \le t_0$. Then

$$\begin{aligned} |u(t)|\,|t|^{\mu}e^{\nu\lambda_n t+\mu\log|t|} &\ge C_4^{1/2}e^{\nu(\lambda_n-\lambda_1/2)t_1+\mu\log|t_1|} - |f|\int_{-\infty}^{t_1} e^{\nu(\lambda_n-\lambda_1/4)\tau}\,d\tau \\ &= C_4^{1/2}e^{\nu(\lambda_n-\lambda_1/2)t_1+\mu\log|t_1|} - |f|\frac{e^{\nu(\lambda_n-\lambda_1/4)t_1}}{\nu(\lambda_n-\lambda_1/4)} \end{aligned}$$

if $t \le t_1 \le t_0$. The last expression is positive for small enough t_1, and (4.10) follows. □

5. Eulerian dynamics on the invariant sets $\mathcal{M}_n$

We will consider certain weak solutions of the Euler equation $\dot{u}+B(u,u) = 0$. A function u is a weak solution of the Euler equation on an interval I if $u \in L^\infty_{\mathrm{loc}}(I,V)\cap C(I,V')$ and

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} B(u(\tau),u(\tau))\,d\tau , \qquad t_1,t_2 \in I \tag{5.1}$$

in V'. Note that $u \in L^\infty_{\mathrm{loc}}(I,V)$ implies the existence of the integral in V'. Therefore, (5.1) implies that $u\colon I \to V'$ is locally absolutely continuous. Using the Galerkin approximation, one obtains the following existence theorem: For every initial datum $u_0 \in V$ there exists a solution u of the Euler equation on $I = \mathbb{R}$ such that $u(0) = u_0$ and

$$\left|\int_0^t \|u(\tau)\|^2\,d\tau\right| \le |t|\,\|u_0\|^2$$

for every $t \in \mathbb{R}$. Also, one can prove the following statement (*see* also [CET]):

LEMMA 5.1. – *If u is a solution of the Euler equation on an interval I, then $|u(t_1)| = |u(t_2)|$ for all $t_1,t_2 \in I$.*

Proof. – Let $t_1, t_2 \in I$, and let $t_2 > t_1$. For every n, $P_n u$ is a locally absolutely continuous function with values in H. Hence, by (2.7),

$$\begin{aligned} |P_n u(t_2)|^2 - |P_n u(t_1)|^2 &= \int_{t_1}^{t_2} \frac{d}{d\tau}\left|P_n u(\tau)\right|^2 d\tau \\ &= -2\int_{t_1}^{t_2} \big(B(u(\tau),u(\tau)),P_n u(\tau)\big)\,d\tau \\ &= 2\int_{t_1}^{t_2} \big(B(u(\tau),u(\tau)),(I-P_n)u(\tau)\big)\,d\tau. \end{aligned}$$

Due to (2.5), the absolute value of the last expression is dominated by

$$2C \int_{t_1}^{t_2} |u(\tau)|^{1/2}\|u(\tau)\|^{3/2}|(I-P_n)u(\tau)|^{1/2}\|(I-P_n)u(\tau)\|^{1/2}\,d\tau$$
$$\leq \frac{2C}{\lambda_{n+1}^{1/4}} \int_{t_1}^{t_2} |u(\tau)|^{1/2}\|u(\tau)\|^{5/2}\,d\tau.$$

Since $u \in L^\infty_{\text{loc}}(I, V)$, the last integral is finite, and as we let $n \to \infty$, we obtain the desired assertion. □

In this section we will show that the dynamics on $\mathcal{M}_n$, for any fixed n, is approximately Eulerian. Let $u(t)$, where $t \in \mathbb{R}$, be a solution of the NSE for which $u(0) = u_0 \in \mathcal{M}_n \backslash \mathcal{A}$. Then $v = u/|u|$ satisfies

$$(5.2) \qquad \dot{v} = -\nu(A - \|v\|^2)v - |u|B(v,v) + \frac{1}{|u|}(f - (f,v)v).$$

Using the rescaled time

$$T = \frac{1}{\nu}\int_0^t |u(\tau)|\,d\tau\ , \qquad t \in \mathbb{R}$$

we denote

$$\eta(T) = |u(t)|\ , \qquad T \in \mathbb{R}$$

and

$$\zeta(T) = \nu v(t)\ , \qquad T \in \mathbb{R}.$$

With these rescalings, ζ has still the dimension of a velocity and T that of time. The equation (5.2) then becomes

$$(5.3) \qquad \frac{d}{dT}\zeta + B(\zeta,\zeta) = F(T),$$

where

$$F(T) = -\frac{1}{\eta}(\nu^2 A - \|\zeta\|^2)\zeta + \frac{1}{\eta^2}(\nu^2 f - (f,\zeta)\zeta).$$

In the sequel we will show that the forcing term $F(T)$ in (5.3) satisfies a certain smallness condition as $T \to -\infty$ (*see* (5.7) below), which will imply the following statement:

THEOREM 5.2. – *Let $\{T_j\}_{j=1}^\infty \subseteq \mathbb{R}$ be a sequence such that $\lim_{j\to\infty} T_j = -\infty$, and let $\zeta_j(T) = \zeta(T + T_j)$ for $T \in \mathbb{R}$. There exists a subsequence $\{T_{n_j}\}_{j=1}^\infty$ of $\{T_j\}_{j=1}^\infty$ such that $\{\zeta_{n_j}\}_{j=1}^\infty$ converges in $L^2_{\text{loc}}(\mathbb{R}, H)$ and in $C_{\text{loc}}(\mathbb{R}, V')$ to a nonzero global solution ζ_∞ of the Euler equation (5.1).*

The statement immediately implies that the convergence of the subsequence also takes place in $C_{\text{loc}}(\mathbb{R}, H_{\text{weak}})$, where H_{weak} is the space H equipped with the weak topology.

Moreover, it will be clear from the proof that $\{\zeta_j\}_{j=1}^\infty$ is a bounded sequence in $C([-T_0, T_0], V')$ for every $T_0 > 0$. Hence, by Aubin's theorem, we may by passing to a subsequence assume that the asserted convergence in the theorem holds in $L^p_{\mathrm{loc}}(\mathbb{R}, H)$ for all $p \in [1, \infty)$.

Proof. – Without loss of generality, we may assume that $u(t) \neq 0$ for $t \leq 0$. Let $\{T_j\}_{j=1}^{\infty}$ and $\{\zeta_j\}_{j=1}^{\infty}$ be as in the statement. We will first show that given any $T_0 > 0$ the sequences of functions $\{\|\zeta_j(T)\|\}_{j=1}^{\infty}$ and $\{|A^{-1/2}(d/dT)\zeta_j(T)|\}_{j=1}^{\infty}$ are uniformly bounded for $T \in [-T_0, T_0]$. The first sequence is uniformly bounded due to $\lim_{T\to-\infty} \|\zeta(T)\|^2 \in \{\nu^2\lambda_1, \dots, \nu^2\lambda_n\}$. Regarding the second sequence, we have by virtue of (5.3) and (2.6)

$$\left|A^{-1/2}\frac{d}{dT}\zeta_j(T)\right| \leq C|\zeta_j|\,\|\zeta_j\| + \frac{\nu^2}{\eta}\|\zeta_j\| + \frac{1}{\eta}\|\zeta_j\|^2|A^{-1/2}\zeta_j| + \frac{\nu^2}{\eta^2}|A^{-1/2}f| + \frac{1}{\eta^2}|A^{-1/2}f|\,\|\zeta_j\|\,|A^{-1/2}\zeta_j|$$

for all $T \in \mathbb{R}$. Since $\lim_{T\to-\infty}\eta(T) = \infty$ as $u_0 \notin \mathcal{A}$, and since $|A^{-1/2}\zeta(T)| \leq (1/\lambda_1^{1/2})|\zeta(T)| = \nu/\lambda_1^{1/2}$ for all T (except where $u(T) = 0$), we get the desired assertion. Now, by [CF, Lemma 8.4], we may assume by passing to a subsequence that $\{\zeta_j\}_{j=1}^{\infty}$ converges to ζ_∞ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}, H)$ and $C_{\mathrm{loc}}(\mathbb{R}, V')$ and weakly in $L^2_{\mathrm{loc}}(\mathbb{R}, V)$. Clearly, $\zeta_\infty \in L^\infty_{\mathrm{loc}}(\mathbb{R}, H) \cap C(\mathbb{R}, V')$. It remains to prove that ζ_∞ satisfies

$$\zeta_\infty(\tau_2) - \zeta_\infty(\tau_1) = -\int_{\tau_1}^{\tau_2} B\big(\zeta_\infty(T), \zeta_\infty(T)\big)\, dT \tag{5.4}$$

for all $\tau_1, \tau_2 \in \mathbb{R}$ such that $\tau_2 \geq \tau_1$. Let $j \in \mathbb{N}$, and fix $\tau_1, \tau_2 \in \mathbb{R}$ for which $\tau_2 \geq \tau_1$. Then

$$\zeta(\tau_2 + T_j) - \zeta(\tau_1 + T_0) = \int_{\tau_1}^{\tau_2} F(T + T_j)\, dT - \int_{\tau_1}^{\tau_2} B\big(\zeta(T + T_j), \zeta(T + T_j)\big)\, dT.$$

We will establish (5.4) by showing that

$$\lim_{j\to\infty} \int_{\tau_1}^{\tau_2} |F(T + T_j)|\, dT = 0 \tag{5.5}$$

and

$$\lim_{j\to\infty} \int_{\tau_1}^{\tau_2} B\big(\zeta(T + T_j), \zeta(T + T_j)\big)\, dT = \int_{\tau_1}^{\tau_2} B\big(\zeta_\infty(T), \zeta_\infty(T)\big)\, dT, \tag{5.6}$$

the limit being taken in V'. First, we have

$$\begin{aligned}\int_{-\infty}^{0} |F(T)|^2\eta(T)\, dT &\leq 2\int_{-\infty}^{0} \frac{1}{\eta}|(\nu^2 A - \|\zeta\|^2)\zeta|^2\, dT \\ &\quad + 2\int_{-\infty}^{0} \frac{1}{\eta^3}|\nu^2 f - (f, \zeta)\zeta|^2\, dT \\ &\leq 2\nu^5 \int_{-\infty}^{0} |(A - \|v\|^2)v|^2\, dt + 8|f|^2\nu^3 \int_{-\infty}^{0} \frac{dt}{|u(t)|^2} < \infty.\end{aligned} \tag{5.7}$$

(Theorem 4.2 implies that the second term is finite, while by integrating (3.4), we get that the same holds for the first term.) Then, for any $j \in \mathbb{N}$,

$$\int_{\tau_1}^{\tau_2} |F(T+T_j)|\,dT = \int_{\tau_1-T_j}^{\tau_2-T_j} |F(T)|\,dT$$
$$\leq \left(\int_{\tau_1-T_j}^{\tau_2-T_j} |F(T)|^2\eta(T)\,dT\right)^{1/2}\left(\int_{\tau_1-T_j}^{\tau_2-T_j} \frac{dT}{\eta(T)}\right)^{1/2}.$$

As $j \to \infty$, the first factor converges to 0 because of (5.7), while the second term converges to 0 since $\lim_{T\to-\infty}\eta(T) = \lim_{t\to-\infty}|u(t)| = \infty$. This gives (5.5).

As for (5.6), we get with a help of (2.6),

$$\int_{\tau_1}^{\tau_2} \left|A^{-1/2}B\big(\zeta(T+T_j),\zeta(T+T_j)\big) - A^{-1/2}B\big(\zeta_\infty(T),\zeta_\infty(T)\big)\right| dT$$
$$\leq C\int_{\tau_1}^{\tau_2} |\zeta(T+T_j) - \zeta_\infty(T)|^{1/2}\,\|\zeta(T+T_j)$$
$$- \zeta_\infty(T)\|^{1/2}\,|\zeta(T+T_j)|^{1/2}\,\|\zeta(T+T_j)\|^{1/2}\,dT$$
$$+ C\int_{\tau_1}^{\tau_2} |\zeta_\infty(T)|^{1/2}\,\|\zeta_\infty(T)\|^{1/2}\,|\zeta(T+T_j)$$
$$- \zeta_\infty(T)|^{1/2}\,\|\zeta(T+T_j) - \zeta_\infty(T)\|^{1/2}\,dT$$

and (5.6), the limit being taken in V', follows immediately. □

For every function $f\colon(-\infty,0] \to V'$, denote by

$$\alpha(f) = \Big\{u_0 \in H : \text{ there exists a sequence } T_1 > T_2 > \cdots \text{ with}$$
$$\lim_{j\to\infty} T_j = -\infty \text{ such that } \lim_{j\to\infty} |A^{-1/2}(f(T_j) - u_0)| = 0\Big\}$$

its α-limit set. Also, for every n, we introduce the sets

$$\mathcal{A}_n = \bigcup_{u_0\in\mathcal{M}_n\setminus\mathcal{A}} \alpha\left(\frac{S(\cdot)u_0}{|S(\cdot)u_0|}\right).$$

This set plays a role of the global attractor for the dynamics of the normalized solutions $S(t)u_0/|S(t)u_0|$ for $u_0 \in \mathcal{M}_n\setminus\mathcal{A}$ as $t \to -\infty$. We justify this with the following theorem which also underlines the connection with the Euler equation.

THEOREM 5.3. – *(i)* $\mathcal{A}_n$ *is a relatively compact subset (in* H*) of* $\{u_0 \in H : |u_0| = 1\}$*;*

(ii) $\lim_{t\to\infty} \operatorname{dist}_{V'}(S(t)u_0/|S(t)u_0|, \mathcal{A}_n) = 0$ *where* $\operatorname{dist}_{V'}$ *denotes the distance in* V' *from a point to a set;*

(iii) $\mathcal{A}_n \subseteq V$ *and* $\|u_0\| \leq \lambda_n$ *for all* $u_0 \in \mathcal{A}_n$*;*

(iv) for every $u_0 \in \mathcal{A}_n$*, there exists a global solution* u *of the Euler equation such that* $u(t) \in \mathcal{A}_n$ *for every* $t \in \mathbb{R}$.

Regarding (iv), we remark that it is not known whether solutions of the Euler equation are unique. Have we had uniqueness, (iv) would state that $\mathcal{A}_n$ is invariant under the flow generated by the Euler equation.

Proof. – The theorem follows easily from Theorem 5.2 and Remark 3.8. □

Note that if $u_0 \in H$ is an eigenfunction of A, the constant function $u(t) = u_0$, for every $t \in \mathbb{R}$, is a solution of the Euler equation, due to $B(u_0, u_0) = 0$. Moreover, this solution is unique among the solutions starting at u_0 because of its regularity. This fact and Theorem 5.2 give the next theorem.

THEOREM 5.4. – *Let $u(t)$, for $t \in \mathbb{R}$, be an arbitrary solution of the NSE such that $u(0) = u_0 \in \mathcal{M}_n \backslash \mathcal{A}$. With the notation as in Theorem 5.2, there exists a sequence $\{T_j\}_{j=1}^{\infty}$ such that $\{\zeta_j\}_{j=1}^{\infty}$ converges in $L^2_{\mathrm{loc}}(\mathbb{R}, H)$ and in $C_{\mathrm{loc}}(\mathbb{R}, V')$ to a stationary solution ζ_∞ of the Euler equation (5.1) which is an eigenfunction of A.*

Clearly, this implies that $P_n H \cap \alpha(S(\cdot)u_0/|S(\cdot)u_0|) \neq \emptyset$ for all $u_0 \in \mathcal{M}_n \backslash \mathcal{A}$ and thus also $\mathcal{A}_n \cap P_n H \neq \emptyset$.

Proof. – Choose $t_0 \in \mathbb{R}$ such that $u(t) \neq 0$ for every $t \leq t_0$. Then the definition of $\mathcal{M}_n$ implies $\sup_{t\in(-\infty,t_0]} \|v(t)\| < \infty$, and thus by (3.4) we get

$$\int_0^{\infty} \left|(A - \|v(t)\|^2)v(t)\right|^2 dt < \infty.$$

We can therefore choose a sequence $t_1 > t_2 > \cdots$ with $\lim_{j\to\infty} t_j = -\infty$ such that

$$\lim_{j\to\infty} \left|(A - \|v(t_j)\|^2)v(t_j)\right| = 0.$$

Now, note that $\lim_{j\to\infty} \|v(t_j)\|^2 = \lambda_k$ for some $k \in \{1, \ldots, n\}$; hence,

$$\lim_{j\to\infty} |(A - \lambda_k)v(t_j)| = 0.$$

Since $\sup_{j\in\mathbb{N}} \|v(t_j)\| < \infty$, we may by passing to a subsequence assume that $\lim_{j\to\infty} |v(t_j) - v_0| = 0$ for some $v_0 \in V$. Since A is closed, we obtain $Av_0 = \lambda_k v_0$.

Now, let

$$T_j = \frac{1}{\nu} \int_0^{t_j} |u(\tau)|\, d\tau\ , \qquad j \in \mathbb{N}.$$

Using Theorem 5.2, we may, by passing to a subsequence, assume that ζ_j converge to a global solution of the Euler equation. Clearly, $\zeta_\infty(0) = v_0$, and by uniqueness we get $\zeta_\infty(t) = v_0$ for every $t \in \mathbb{R}$. □

For an illustrative example, we return once again to the equation

$$\dot{u} + \nu A u = f$$

discussed in Sections 1 and 3. All the constructions in this paper apply also to this case—we only have to take everywhere $B = 0$. Recall that for each n

$$\mathcal{M}_n^{\mathrm{lin}} = \left\{ u_0 \in H : u_0 - \frac{1}{\nu} A^{-1} f \in P_n H \right\}$$

is the invariant manifold which is the analog of $\mathcal{M}_n$. Likewise, let $\mathcal{A}^{\text{lin}} = \{A^{-1}f/\nu\}$ and $\mathcal{A}_n^{\text{lin}}$ be the counterparts of $\mathcal{A}$ and $\mathcal{A}_n$ respectively. The role of the Euler equation is played by $\dot{u} = 0$. We are able to compute $\mathcal{A}_n^{\text{lin}}$ here. Let $u_0 \in \mathcal{M}_n^{\text{lin}} \backslash \mathcal{A}_n^{\text{lin}}$. Then $u_0 - A^{-1}f/\nu = p_0$ for some nonzero $p_0 \in P_n H$. Since $Ap_0 = \lambda_k p_0$ for some $k \in \{1, \dots, n\}$, we get

$$\lim_{t\to\infty} \frac{S^{\text{lin}}(t)u_0}{|S^{\text{lin}}(t)u_0|} = \lim_{t\to\infty} \frac{\nu e^{-\nu\lambda_k t}p_0 + A^{-1}f}{|\nu e^{-\nu\lambda_k t}p_0 + A^{-1}f|} = \frac{p_0}{|p_0|}.$$

Thus,

$$\alpha\left(\frac{S^{\text{lin}}(\cdot)u_0}{|S^{\text{lin}}(\cdot)u_0|}\right) = \left\{\frac{p_0}{|p_0|}\right\}$$

and we obtain

$$\mathcal{A}_n^{\text{lin}} = \left\{\frac{p_0}{|p_0|} : p_0 \in P_n H \backslash \{0\}\right\}$$

which is a compact subset of $\{u_0 \in H : |u_0| = 1\}$ (compare this with Theorem 5.3(iii)). It is not clear whether $\mathcal{A}_n = \mathcal{A}_n^{\text{lin}}$ for any n.

6. A transitivity type property of the invariant sets $\mathcal{M}_n$

Theorem 4.1 implies $P_n\mathcal{G} = P_n H$ for all n. We will show in this section that $P\mathcal{G} = PH$ for every orthogonal projector P in H which satisfies $\dim PH < \infty$ and $PH \subseteq V$. More generally:

THEOREM 6.1. – *Let $T: V' \to V$ be a finite rank operator. Then there exists $n \in \mathbb{N}$ such that $T\mathcal{M}_n = TH$.*

Remark 6.2. – (i) It will be clear from the proof that there are in fact infinitely many n with this property.

(ii) The theorem implies $T\mathcal{G} = TH$.

(iii) Theorem 6.1 also implies that given any m independent vectors $v_1, \dots, v_m \in V$ (m arbitrary) and any m numbers $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ there exists $x \in \mathcal{M}_n$ for some n such that

$$(x, v_j) = \alpha_j , \qquad j = 1, \dots, m. \qquad \square$$

The proof of Theorem 6.1 is a generalization of the argument used in the proof of (4.1). However, before the proof we need to establish some technical facts.

Let T be an operator as in the above statement. Then

$$Tx = (x, g_1)f_1 + \cdots + (x, g_k)f_k$$

for some linearly independent $f_1, \dots, f_k \in V$ and some linearly independent $g_1, \dots, g_k \in V$ ($k = \dim TH$). We only have to consider the case $TH = T^*H$, *i.e.*, $\mathcal{L}\{f_1, \dots, f_k\} = \mathcal{L}\{g_1, \dots, g_k\}$, where $\mathcal{LS}$ denotes the set of linear combinations of elements from a set $\mathcal{S}$. Indeed, if $TH \neq T^*H$, we can find a finite rank operator $\widetilde{T}$ with $\widetilde{T}H = \widetilde{T}^*H \subseteq V$

such that $P\widetilde{T} = T$ for some orthogonal projector P; then $\widetilde{T}\mathcal{M}_n = \widetilde{T}H$ clearly implies $T\mathcal{M}_n = TH$.

Another observation is that there exists $\delta > 0$ such that

$$\|f\| \leq \delta |f|^2 , \qquad f \in TH. \tag{6.1}$$

This is because all norms are equivalent on a finite dimensional subspace TH. Also, for every n, we have

$$|(I - P_n)f| \leq \frac{1}{\lambda_{n+1}^{1/2}} \|(I - P_n)f\| \leq \frac{1}{\lambda_{n+1}^{1/2}} \|f\| \leq \left(\frac{\delta}{\lambda_{n+1}}\right)^{1/2} |f| , \qquad f \in TH \tag{6.2}$$

whence

$$|P_n f| \geq \left(1 - \left(\frac{\delta}{\lambda_{n+1}}\right)^{1/2}\right) |f| , \quad f \in TH. \tag{6.3}$$

For any $n \in \mathbb{N}$ for which $m = m(n) = \dim P_n H > k$, choose $x_{k+1}^{(n)}, \ldots, x_m^{(n)} \in P_n H$ ($m = m(n) = \dim P_n H$) so that

$$\left(x_i^{(n)}, x_j^{(n)}\right) = \delta_{ij} , \qquad i, j \in \{k+1, \ldots, m\}$$

and

$$\left(x_i^{(n)}, f_j\right) = 0 , \qquad i \in \{k+1, \ldots, m\} , \quad j \in \{1, \ldots, k\}.$$

Let also

$$T_n x = (x, g_1) f_1 + \cdots + (x, g_k) f_k + \left(x, x_{k+1}^{(n)}\right) x_{k+1}^{(n)} + \cdots + \left(x, x_m^{(n)}\right) x_m^{(n)} , \qquad x \in H$$

and recall that

$$\mathcal{C}_n = \left\{ x \in V : \|x\|^2 \leq \frac{\lambda_n + \lambda_{n+1}}{2} |x|^2 \right\}$$

for all n.

LEMMA 6.3. – *There exist $n \in \mathbb{N}$ and $\epsilon > 0$ such that*

(i) $\operatorname{Ker} T_n \cap \mathcal{C}_n = \{0\}$;

(ii) $|T_n x| \geq \epsilon |x|$ *for* $x \in \mathcal{C}_n$;

(iii) $T_n H = \mathcal{L}\left\{f_1, \ldots, f_k, x_{k+1}^{(n)}, \ldots, x_m^{(n)}\right\} \subseteq \mathcal{C}_n$.

Proof of Lemma 6.3. – (i) Let $x_0 \in \operatorname{Ker} T_n \cap \mathcal{C}_n$, for any fixed n such that $m = \dim P_n H > k$. Then

$$(x_0, x) = 0 , \qquad x \in \mathcal{L}\left\{g_1, \ldots, g_k, x_{k+1}^{(n)}, \ldots, x_m^{(n)}\right\} \tag{6.4}$$

and in particular

$$(6.5) \qquad (x_0, x) = 0\,, \qquad x \in TH.$$

We choose $x_1^{(n)}, \ldots, x_k^{(n)}$ such that $x_1^{(n)}, \ldots, x_m^{(n)}$ is an orthonormal basis of P_nH. First, we claim that if

$$(6.6) \qquad \delta < \lambda_{n+1},$$

then

$$(6.7) \qquad P_nTH = \mathcal{L}\Big\{x_1^{(n)}, \ldots, x_k^{(n)}\Big\}.$$

Let $x \in P_nTH$. Then $x = P_n(\alpha_1 f_1 + \cdots + \alpha_k f_k)$ for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Therefore,

$$\Big(x, x_i^{(n)}\Big) = \Big(P_n(\alpha_1 f_1 + \cdots + \alpha_k f_k), x_i^{(n)}\Big) = \Big(\alpha_1 f_1 + \cdots + \alpha_k f_k, x_i^{(n)}\Big) = 0$$

for $i = k+1, \ldots, m$, whence $x \in \mathcal{L}\Big\{x_1^{(n)}, \ldots, x_k^{(n)}\Big\}$. This shows $P_nTH \subseteq \mathcal{L}\Big\{x_1^{(n)}, \ldots, x_k^{(n)}\Big\}$. Since $\mathcal{L}\Big\{P_nf_1^{(n)}, \ldots, P_nf_k^{(n)}\Big\} = P_nTH$, it remains to prove $\operatorname{Ker} P_n \cap TH = \{0\}$. Let $f \in \operatorname{Ker} P_n \cap TH$. Then

$$\lambda_{n+1}|f|^2 = \lambda_{n+1}|(I - P_n)f|^2 \le \|(I-P_n)f\|^2 = \|f\|^2 \le \delta|f|^2$$

by virtue of (6.1). Because of (6.6), we conclude $f = 0$, and (6.7) is established.

Assuming (6.6), and thus (6.7), there exist $f_i^{(n)} \in TH$ $(i = 1, 2, \ldots, k)$ such that $P_nf_i^{(n)} = x_i^{(n)}$. Then

$$\begin{aligned}
\Big|\Big(x_0, x_i^{(n)}\Big)\Big| &= \Big|\Big(x_0, P_nf_i^{(n)}\Big)\Big| = \Big|\Big(x_0, (I-P_n)f_i^{(n)}\Big)\Big| \\
&\le |x_0|\Big|(I-P_n)f_i^{(n)}\Big| \le \left(\frac{\delta}{\lambda_{n+1}}\right)^{1/2}\Big|f_i^{(n)}\Big|\,|x_0| \\
&\le \frac{(\delta/\lambda_{n+1})^{1/2}}{1-(\delta/\lambda_{n+1})^{1/2}}\Big|P_nf_i^{(n)}\Big|\,|x_0| = \frac{(\delta/\lambda_{n+1})^{1/2}}{1-(\delta/\lambda_{n+1})^{1/2}}|x_0| \\
&= \frac{\delta^{1/2}}{\lambda_{n+1}^{1/2} - \delta^{1/2}}|x_0|
\end{aligned}$$

for every $i \in \{1, 2, \ldots, k\}$, where we used (6.5), (6.2), and (6.3). Together with (6.4), this implies

$$|P_nx_0| = \left|\sum_{i=1}^m \Big(x_0, x_i^{(n)}\Big)x_i^{(n)}\right| = \left|\sum_{i=1}^k \Big(x_0, x_i^{(n)}\Big)x_i^{(n)}\right| \le k\frac{\delta^{1/2}}{\lambda_{n+1}^{1/2} - \delta^{1/2}}|x_0|.$$

Since also $x_0 \in \mathcal{C}_n$, Lemma 3.4 implies

$$|x_0| \le \left(\frac{2\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right)^{1/2}|P_nx_0| \le k\left(\frac{2\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right)^{1/2}\frac{\delta^{1/2}}{\lambda_{n+1}^{1/2} - \delta^{1/2}}|x_0|.$$

Therefore, if

$$k\left(\frac{2\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^{1/2}\frac{\delta^{1/2}}{\lambda_{n+1}^{1/2}-\delta^{1/2}}<1 \tag{6.8}$$

and if (6.6) is satisfied (according to (2.3) and (2.4), this happens for infinitely many n), we get $x_0 = 0$ as desired.

(ii) Fix any n for which (i) holds. Suppose that there is no $\epsilon > 0$ so that (ii) holds; then there exist $x_1, x_2, \ldots \in \mathcal{C}_n$ such that

$$|x_i| = 1\ , \qquad i \in \mathbb{N} \tag{6.9}$$

and

$$\lim_{i\to\infty} |T_n x_i| = 0. \tag{6.10}$$

Thus, also $\|x_i\| \leq (\lambda_n + \lambda_{n+1})/2$. Passing to a subsequence, we may assume $\lim_{i\to\infty}|x_i - x_0| = 0$ for some $x_0 \in V$ (*see* Remark 3.8). But then (6.9) and (6.10) imply $|x_0| = 1$ and $T_n x_0 = 0$. Since $\mathcal{C}_n$ is closed in H (*see* Remark 3.8), $x_0 \in \mathcal{C}_n$, and therefore $x_0 = 0$ by (i). We arrived to a contradiction and hence established (ii).

(iii) Fix any n for which

$$\lambda_{n+1} - \lambda_n \geq 2\delta \tag{6.11}$$

and let $\alpha_1, \ldots, \alpha_m$ be arbitrary. Denote $f = \alpha_1 f_1 + \cdots + \alpha_k f_k$ and $x = \alpha_{k+1}x_{k+1}^{(n)} + \cdots + \alpha_m x_m^{(n)}$. Then for $\epsilon = 2\lambda_n/(\lambda_{n+1} - \lambda_n)$ we get

$$\begin{aligned}
\|f+x\|^2 &\leq \|f\|^2 + 2\|f\|\,\|x\| + \|x\|^2 \leq (1+\epsilon)\|f\|^2 + \left(1+\frac{1}{\epsilon}\right)\|x\|^2 \\
&\leq (1+\epsilon)\delta|f|^2 + \left(1+\frac{1}{\epsilon}\right)\lambda_n|x|^2 = \frac{\lambda_{n+1}+\lambda_n}{\lambda_{n+1}-\lambda_n}\delta|f|^2 + \frac{\lambda_{n+1}+\lambda_n}{2}|x|^2 \\
&\leq \frac{\lambda_{n+1}+\lambda_n}{2}(|f|^2+|x|^2) = \frac{\lambda_{n+1}+\lambda_n}{2}|f+x|^2.
\end{aligned}$$

Hence, (i)–(iii) hold provided n satisfies conditions (6.6), (6.8), and (6.11). □

Now, we can conclude the proof of Theorem 6.1:

Proof of Theorem 6.1. – Fix $n \in \mathbb{N}$ and $\epsilon > 0$ provided by Lemma 6.3. In particular, $T_n H \subseteq \mathcal{C}_n$. First, we shall establish

$$T_n S(t_0) T_n H = T_n H\ , \qquad t_0 > 0. \tag{6.12}$$

Note that this is an analogue of Lemma 3.5 which claims (6.12) with T_n replaced by P_n. The proof of (6.12) is, except for some obvious changes, analogous to the proof of Lemma 3.5. (We substitute P_n with T_n and $1/(\gamma_n + 1)$ with ϵ; also, we use Lemma 6.3(ii) instead of Lemma 3.4.) We omit further details.

Next, let $p_0 \in T_n H$, and choose a sequence $0 > t_1 > t_2 > \cdots$ such that $\lim_{k\to\infty} t_k = -\infty$. By (6.12), there exist sequences $\{u_k\}_{k=1}^{\infty} \subseteq H$ and $\{p_k\}_{k=1}^{\infty} \subseteq T_n H$ such that $S(-t_k)p_k = u_k$ and $T_n u_k = p_0$ for $k \in \mathbb{N}$. Our proof now almost completely follows the proof of Lemma 3.9. Passing to a subsequence, we thus obtain a global solution $S(t)u_\infty$ such that $u_\infty \in \mathcal{M}_n$ and $T_n u_\infty = p_0$. This establishes $T_n\mathcal{M}_n = T_n H$. Now, if P is the orthogonal projector (in H) with the range $\mathcal{L}\{f_1, \ldots, f_k\}$, we get $PT_n = T$. Therefore, $T\mathcal{M}_n = TH$, and this concludes the proof. □

7. Open problems

1. Theorem 4.1 implies that $d_F(\mathcal{M}_n \cap B^H(r)) \geq \max\{\dim P_n H, d_F(\mathcal{A})\}$ for all $r > 0$ and $n \in \mathbb{N}$. Do we actually have $d_F(\mathcal{M}_n \cap B^H(r)) = \max\{\dim P_n H, d_F(\mathcal{A})\}$?

2. Is $\mathcal{M}_n$ for large enough n an exponential attractor for the NSE? (*See* [EFNT] for the definition and properties of exponential attractors.)

3. Is $\mathcal{M}_n$ for large enough n a manifold? Is it an inertial manifold ([FST], [CFNT])?

4. How do the ratios $|A^\alpha u(t)|/|u(t)|$, where $\alpha > 1/2$, behave for solutions $u(t)$ on the sets $\mathcal{M}_n$ as $t \to -\infty$?

5. If $f = 0$, we have by [FS1] and [FS2] a family of invariant manifolds M_n characterized by the behavior of the Dirichlet quotients $\|u(t)\|/|u(t)|$ as $t \to \infty$. That is, for every $n \in \{0\} \cup \mathbb{N}$, we have

$$M_n = \{0\} \bigcup \left\{ u_0 \in H\backslash\{0\} : \lim_{t\to\infty} \frac{\|S(t)u_0\|^2}{|S(t)u_0|^2} \in \{\lambda_{n+1}, \lambda_{n+2}, \ldots\} \right\}.$$

By Lemma 3.2 and Corollary 4.3, $\mathcal{M}_n \subseteq V\backslash M_n$ for all $n \in \mathbb{N}$. What are the precise relations between the sets $\mathcal{M}_1, \mathcal{M}_2, \ldots$ and the manifolds $M_1, M_2, \ldots$?

6. Is the set $\mathcal{G}$, which is by definition the set of all initial data which lead to global solutions, dense in H? Note that a good answer in (4) may lead to a solution in (6).

7. Do any of the properties described in this paper carry over to the NSE with Dirichlet boundary conditions?

Acknowledgements

The work of P. Constantin was supported by NSF DMS-9207080 and DOE DE FG02-92ER-25119, that of C. Foias and I. Kukavica by NSF DMS-9024769, NSF DMS-9623161, DOE DE-FG02-92ER25120, and NAVY N00014-91-J-1140, while that of A. J. Majda by NSF DMS-99301094, ARPA N00014-92-J-1796, ONR N00014-89-J-1044.P00003, and ARO DAAL03-92-G-0010.

REFERENCES

[BT] C. Bardos and L. Tartar, Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines, *Arch. Rational Mech. Anal.*, *50*, 1973, pp. 10-25.

[BV] A. V. Babin and M. I. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.

[CET] P. Constantin and W. E, E. S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, *Commun. Math. Phys*, *165*, 1994, pp. 207-209.

[CF] P. Constantin and C. Foias, Navier-Stokes Equations, Chicago Lectures in Mathematics, Chicago/London, 1988.

[CFNT] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Spectral barriers and inertial manifolds for dissipative partial differential equations, *J. Dynam. Diff. Eq.*, *1*, 1988, pp. 45-73.

[CFT] P. CONSTANTIN, C. FOIAS and R. TEMAM, On the dimension of the attractors in two-dimensional turbulence, *Physica D*, *30*, 1988, pp. 284-296.

[DGHN] C. DOERING, J. D. GIBBON, D. HOLM and B. NICOLAENKO, Low-dimensional behavior in the complex Ginzburg-Landau equation, *Nonlinearity*, *1*, 1988, pp. 279-309.

[E] P. ERDÖS, Some problems in elementary number theory, *Publ. Math. Debrecen*, *2*, 1951, pp. 103-109.

[EFNT] A. EDEN, C. FOIAS, B. NICOLAENKO and R. TEMAM, Exponential Attractors for Dissipative Evolution Equations, J. Wiley & Sons, New York, 1994.

[FS1] C. FOIAS and J. C. SAUT, Limite du rapport de l'enstrophie sur l'énergie pour une solution faible des équations de Navier-Stokes, *C. R. Acad. Sci. Paris, Sér. I Math.*, *298*, 1981, pp. 241-244.

[FS2] C. FOIAS and J. C. SAUT, Asymptotic behavior, as $t \to \infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds, *Indiana Univ. Math. J.*, *33*, 1984, pp. 459-477.

[FST] C. FOIAS, G. R. SELL and R. TEMAM, Inertial manifolds for nonlinear evolutionary equations, J. Diff. Eq., *73*, 1988, pp. 309-353.

[FT] C. FOIAS and R. TEMAM, Some analytic and geometric properties of the solutions of the Navier-Stokes equations, *J. Math. Pures Appl.*, *58*, 1979, pp. 339-368.

[H] J. K. HALE, Asymptotic Behavior of Dissipative Systems, AMS, Providence, 1988.

[K] I. KUKAVICA, On the behavior of the solutions of the Kuramoto-Sivashinsky Equations for negative time, *J. Math. Anal. Appl.*, *166*, 1992, pp. 601-606.

[R] I. RICHARDS, On the gaps between numbers which are sums of two squares, *Adv. In Math.*, *46*, 1982, pp. 1-2.

[T1] R. TEMAM, Navier-Stokes Equations and Nonlinear Functional Analysis, SIAM, Philadelphia, 1983.

[T2] R. TEMAM, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, *Appl. Math. Sciences No. 68*, Springer Publ., New York, 1988.

P. CONSTANTIN
Department of Mathematics,
The University of Chicago,
Chicago, IL 60637

C. FOIAS
Department of Mathematics,
Indiana University,
Bloomington, IN 47405

I. KUKAVICA
Department of Mathematics,
The University of Chicago,
Chicago, IL 60637

A. J. MAJDA
Department of Mathematics,
Princeton University,
Princeton, NJ 08540